

1. [9 marks] Find the following coefficients:

(a)

$$[x^n](1 + 3x^2)^{-5}$$

(b)

$$[x^5](1 + 3x^3)^3(1 - x)^{-2}$$

(c)

$$[x^n](1 - 5x)^{-3}(1 + x)^m \quad (\text{where } m \text{ is a positive integer})$$

(d)

$$[x^{73}](1 + x^2)^{10}(1 - x^4)^2$$

2. [12 marks]

(a) Prove that the generating function for compositions of n in which each part is at least 3, and the number of parts is exactly 100, is

$$\frac{x^{300}}{(1 - x)^{100}}.$$

(b) Prove that the generating function for compositions of n in which each part is at least 3 is

$$\frac{1 - x}{1 - x - x^3}.$$

(c) Prove that the generating function for compositions of n in which each part is at least 3, and the number of parts is at most 100, is

$$\frac{(1 - x)^{101} - x^{303}}{(1 - x)^{100}(1 - x - x^3)}.$$

3. [7 marks] Give decompositions that uniquely create the following sets of 0/1-strings.

(a) All strings with no odd blocks of length greater than 4.

(b) All strings with no occurrence of the substring 0011.

4. [6 marks]

- (a) Find the generating function for the following set of strings: $\{\epsilon, 00\}(\{1\}(\{\epsilon\} \cup \{0\}\{00\}^*))^*$. Write your answer as $\frac{p(x)}{q(x)}$ where p and q are polynomials.
- (b) Prove that the following decomposition does *not* uniquely create the set of all strings where every odd block of 0s is immediately followed by an odd block of 1s. (Show *either* that it does not create them all, *or* that it creates some strings more than once.)

$$\{1\}^* (\{0\}\{00\}^*\{1\}\{11\}^* \cup \{00\}\{00\}^*\{11\}\{11\}^*)^* \{00\}^*$$

5. [10 marks]

- (a) Find the general solution of the recurrence equation

$$b_n = -3b_{n-1} + 4b_{n-2} \quad (n \geq 2)$$

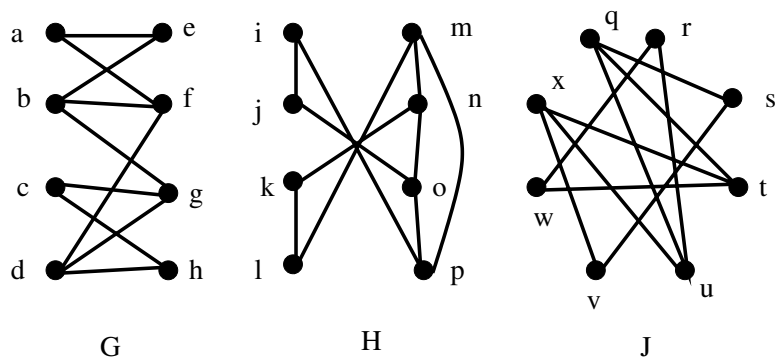
without initial conditions. That is, find constants R and S such that $b_n = A \times R^n + B \times S^n$ for $n \geq 2$, where A and B are constants that would depend on the initial conditions.

- (b) Let the sequence b_n be defined by $b_0 = 1$, $b_1 = 2$, and

$$b_n + 3b_{n-1} - 4b_{n-2} = 5 \quad \text{for all } n \geq 2.$$

Solve this recurrence relation to obtain a closed form expression for b_n .

6. [8 marks] For each of the three pairs (G, H) , (G, J) and (H, J) , determine if the two graphs in the pair are isomorphic. Prove your claim in each case.



7. [8 marks] Let $n \geq 1$ be an integer. The graph H_n is defined as follows: the vertex set of H_n is the set of all strings of 0s, 1s and 2s of length n (these are called *ternary strings*), that is

$$V(H_n) = \{t_1 t_2 \cdots t_n : t_i \in \{0, 1, 2\} \text{ for each } i\}.$$

Two vertices of H_n are joined by an edge if they differ in exactly one coordinate.

- (a) Draw H_1 and H_2 .
- (b) Find the number of vertices of H_n . Prove your answer is correct.
- (c) Find the number of edges of H_n . Prove your answer is correct.