1. **[9 marks]** Find the following coefficients:

(a)

$$[x^n](1+3x^2)^{-5}$$

Solution [2 marks]

$$\begin{aligned} & [x^n](1+3x^2)^{-5} \\ = \quad & [x^n]\textstyle\sum_{k=0}^{\infty}\binom{-5}{k}(-3)^k x^{2k} \\ = \quad & [x^n]\textstyle\sum_{k=0}^{\infty}\binom{4+k}{k}(-3)^k x^{2k} \\ = \quad & \begin{cases} \binom{4+n/2}{n/2}(-3)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

ALTERNATIVE:

$$\begin{cases} \binom{4+n/2}{4}(-3)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

(b)

$$[x^5](1+3x^3)^3(1-x)^{-2}$$

Solution [2 marks]

$$\begin{aligned} & [x^5](1+3x^3)^3(1-x)^{-2} \\ = \quad & [x^5](1+3\cdot 3x^3+\ldots)(1+2x+3x^2+4x^3+5x^4+6x^5+\ldots) \\ = \quad & 6+9\cdot 3 = 33 \end{aligned}$$

(c)

$$[x^n](1-5x)^{-3}(1+x)^m \quad \text{(where } m \text{ is a positive integer)}$$

Solution [3 marks]

$$\begin{aligned} & [x^n](1-5x)^{-3}(1+x)^m \\ = \quad & [x^n]\left(\textstyle\sum_{i=0}^{\infty}\binom{i+2}{2}5^i x^i\right)\left(\textstyle\sum_{j=0}^{\infty}\binom{m}{j}x^j\right) \\ = \quad & \textstyle\sum_{j=0}^{m}\binom{n-j+2}{2}5^{n-j}\binom{m}{j} \end{aligned}$$

(the summation could also be refined further and let j go from 0 to $\min(m,n)$)
Note that there are alternative ways of expressing the final answer, e.g.

$$\sum_{i=0}^{n}\binom{i+2}{2}5^i\binom{m}{n-i}$$

(in this case, the summation could also be refined further and let i go from $\max(0, n-m)$ to n)

(d)

$$[x^{73}](1+x^2)^{10}(1-x^4)^2$$

Solution [2 marks]

$$[x^{73}](1+x^2)^{10}(1-x^4)^2 = 0$$

since $(1+x^2)^{10}$ and $(1-x^4)^2$ are both power series in x^2, and thus their product is a power series in x^2, which implies that there are no odd powers of x.

2. **[12 marks]**

(a) Prove that the generating function for compositions of n in which each part is at least 3, and the number of parts is exactly 100, is

$$\frac{x^{300}}{(1-x)^{100}}.$$

Solution [**4 marks**] Using the definition of generating function, we find the generating function for $N_{\geq 3} = \{3, 4, 5, \ldots\}$ is $\Phi_{N_{\geq 3}} = x^3 + x^4 + x^5 + \ldots = x^3(1 + x + x^2 + \ldots) = \frac{x^3}{1-x}$.
The set of compositions S can be described as a Cartesian product

$$S = N_{\geq 3} \times N_{\geq 3} \times \ldots \times N_{\geq 3} = N_{\geq 3}^{100}.$$

Therefore by the Product Lemma we find

$$\Phi_S = \Phi_{N_{\geq 3}}^{100} = (\frac{x^3}{1-x})^{100} = \frac{x^{300}}{(1-x)^{100}}.$$

(b) Prove that the generating function for compositions of n in which each part is at least 3 is

$$\frac{1-x}{1-x-x^3}.$$

Solution [**4 marks**] As above we know $\Phi_{N_{\geq 3}} = \frac{x^3}{1-x}$. The set of compositions S can be described as a union of Cartesian products

$$S = \bigcup_{k \geq 0} N_{\geq 3}^k.$$

As above, each $\Phi_{N_{\geq 3}^k} = (\frac{x^3}{1-x})^k$. Thus by the Sum Lemma

$$\Phi_S = \sum_{k \geq 0} \Phi_{N_{\geq 3}^k} = \sum_{k \geq 0} (\frac{x^3}{1-x})^k = \frac{1}{1 - \frac{x^3}{1-x}}.$$

Simplifying this last expression gives

$$\Phi_S = \frac{1-x}{1-x-x^3}.$$

(c) Prove that the generating function for compositions of n in which each part is at least 3, and the number of parts is at most 100, is

$$\frac{(1-x)^{101} - x^{303}}{(1-x)^{100}(1-x-x^3)}.$$

Solution [**4 marks**]
We know that for the finite geometric sum $\sum_{i=0}^{t-1} y^i = \frac{1-y^t}{1-y}$. The required set of compositions can be described as

$$S = \bigcup_{k=0}^{100} N_{\geq 3}^k.$$

Thus by the Sum Lemma

$$\Phi_S = \sum_{k=0}^{100} \Phi_{N_{\geq 3}^k} = \sum_{k=0}^{100} (\frac{x^3}{1-x})^k = \frac{1-(\frac{x^3}{1-x})^{101}}{1-\frac{x^3}{1-x}}.$$

Simplifying this last expression gives

$$\Phi_S = (\frac{(1-x)^{101} - x^{303}}{(1-x)^{101}})(\frac{1-x}{1-x-x^3}) = \frac{(1-x)^{101} - x^{303}}{(1-x)^{100}(1-x-x^3)}.$$

3. [**7 marks**] Give decompositions that uniquely create the following sets of 0/1-strings.

(a) All strings with no odd blocks of length greater than 4.

Solution [**3 marks**] Let S_0 and S_1 be the following set of binary strings:

$$S_0 = (\{00\}\{00\}^*) \cup \{0, 000\}, \qquad S_1 = (\{11\}\{11\}^*) \cup \{1, 111\}.$$

Then these are the nonempty strings with no odd blocks of length greater than 4. So the answer is

$$(\{\varepsilon\} \cup S_1)(S_0 S_1)^*(\{\varepsilon\} \cup S_0).$$

An alternative answer is

$$(\{\varepsilon\} \cup S_0)(S_1 S_0)^*(\{\varepsilon\} \cup S_1).$$

(b) All strings with no occurrence of the substring 0011.

Solution [**4 marks**] Start with the standard block decomposition $\{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$. We only need to make sure that the 0-blocks (except the last) that have length at least 2 are not followed by a 1-block of length at least 2. So we only need to restrict the repeating part of the decomposition by subtracting the case that a block of at least two 0's is immediately followed by followed by a block of at least two 1's. So the answer is

$$\{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^* \setminus \{00\}\{0\}^*\{11\}\{1\}^*)^*\{0\}^*.$$

4. [**6 marks**]

(a) Find the generating function for the following set of strings: $\{\epsilon, 00\}(\{1\}(\{\epsilon\} \cup \{0\}\{00\}^*))^*$. Write your answer as $\frac{p(x)}{q(x)}$ where p and q are polynomials.

Solution [**4 marks**]

$$\frac{1+x^2}{1-x(1+x/(1-x^2))}$$
$$= \frac{1-x^4}{1-x-2x^2+x^3}.$$

(b) Prove that the following decomposition does *not* uniquely create the set of all strings where every odd block of 0s is immediately followed by an odd block of 1s. (Show *either* that it does not create them all, *or* that it creates some strings more than once.)

$$\{1\}^* (\{0\}\{00\}^*\{1\}\{11\}^* \cup \{00\}\{00\}^*\{11\}\{11\}^*)^* \{00\}^*$$

Solution [**2 marks**] This decomposition does not create all such strings. It is enough to show that one such string is missed. The string 001 is not created. To be created, the block of zeros has to be part of the $\{00\}\{00\}^*$ set of blocks, but each such block is immediately followed by an even length block of 1s.

5. **[10 marks]**

(a) Find the general solution of the recurrence equation

$$b_n = -3b_{n-1} + 4b_{n-2} \ (n \geq 2)$$

without initial conditions. That is, find constants R and S such that $b_n = A \times R^n + B \times S^n$ for $n \geq 2$, where A and B are constants that would depend on the initial conditions.

Solution [3 marks] The characteristic polynomial is

$$x^2 + 3x - 4 = (x+4)(x-1)$$

with roots -4 and 1, so

$$b_n = A(-4)^n + B(1)^n$$

for constants A and B. Thus $R = -4$ and $S = 1$ (or vice versa).

(b) Let the sequence b_n be defined by $b_0 = 1$, $b_1 = 2$, and

$$b_n + 3b_{n-1} - 4b_{n-2} = 5 \quad \text{for all } n \geq 2.$$

Solve this recurrence relation to obtain a closed form expression for b_n.

Solution [7 marks] We can rewrite the equation as

$$b_n + 3b_{n-1} - 4b_{n-2} = 5 \ (n \geq 2).$$

For a particular solution try $b_n = A$ (A constant). Substituting into the equation, $A + 3A - 4A = 5$, which has no solution. So try $b_n = A + Bn$ (or just $b_n = Bn$ since the As cancel). Then

$$Bn + 3B(n-1) - 4B(n-2) = 5$$

gives $B = 1$. So a particular solution is $b_n = n$. **(3 marks for particular solution)**

The homogeneous equation is solved in part (a) of this question. So the solution is of the form

$$b_n = A(-4)^n + B + n.$$

Putting $n = 0$ and 1 gives

$$b_0 = 1 = A + B,$$

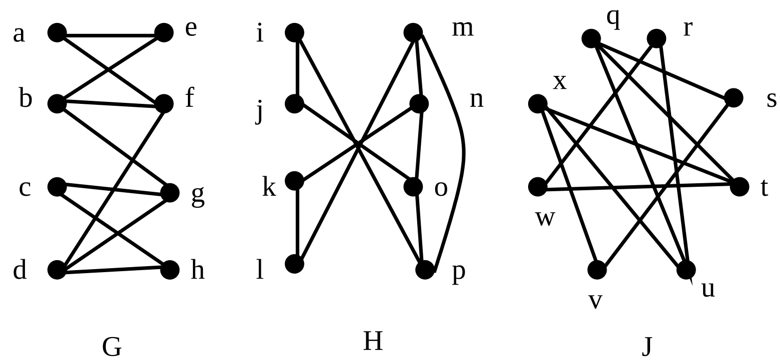
$$b_1 = 2 = -4A + B + 1.$$

Subtracting the equations gives $-5A = 0$ and so $A = 0$. Then $B = 1$. So the final solution is

$$b_n = n + 1$$

for $n \geq 0$.

6. [**8 marks**] For each of the three pairs (G, H), (G, J) and (H, J), determine if the two graphs in the pair are isomorphic. Prove your claim in each case.



Solution

- G **and** H **are isomorphic** since the following one-to-one correspondence between their vertices preserves all the adjacency relations:

v	$f(v)$
a	i
b	o
c	k
d	m
e	j
f	p
g	n
h	l

- G **and** J **are not isomorphic:** Note that J is *not* bipartite, since it has the odd cycle $q{-}s{-}v{-}x{-}u{-}q$. Note that G *is* bipartite (with bipartition $A = \{a, b, c, d\}$ and $B = \{e, f, g, h\}$). Since an isomorphism from G to J would map the bipartition (A, B) to a bipartition of J, there cannot exist an isomorphism. Thus G and J are *not* isomorphic.

- H **and** J **are not isomorphic:** Since G and H are isomorphic, but G and J are not isomorphic, then H and J cannot be isomorphic (it is easy to verify that if $g : V(H) \to V(J)$ is an isomorphism from H to J and $f : V(G) \to V(H)$ is an isomorphism from G to H, then $g \circ f : V(G) \to V(J)$, which maps $v \in V(G)$ to $g(f(v))$, is an isomorphism from G to J.)

  Alternatively, one could directly observe that H and J are not isomorphic, e.g. by noting that H is bipartite but J is not.

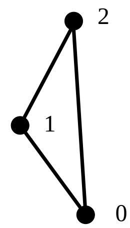
7. **[8 marks]** Let $n \geq 1$ be an integer. The graph H_n is defined as follows: the vertex set of H_n is the set of all strings of of 0s, 1s and 2s of length n (these are called *ternary* strings), that is

$$V(H_n) = \{t_1 t_2 \cdots t_n : t_i \in \{0, 1, 2\} \text{ for each } i\}.$$

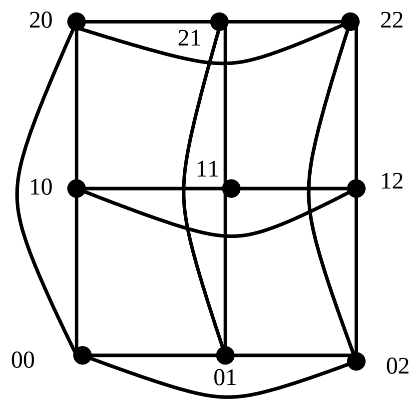
Two vertices of H_n are joined by an edge if they differ in exactly one coordinate.

(a) Draw H_1 and H_2.

Solution [2 marks]



H 1



H 2

(b) Find the number of vertices of H_n. Prove your answer is correct.

Solution [2 marks] Since each position t_i is one of 3 possible values, and the number of positions is n, the number of vertices of H_n is 3^n.

(c) Find the number of edges of H_n. Prove your answer is correct.

Solution [4 marks] Let $v = (t_1, t_2, \ldots, t_n)$ be an arbitrary vertex of H_n. We find the degree of v as follows: each vertex w that is adjacent to v differs from v in exactly one position i. There are therefore n possibilities for i. Now for each i, we could change t_i to two other values (those in $\{0, 1, 2\} \setminus \{t_i\}$) to obtain such a w. Therefore v has degree $2n$ in H_n. Thus H_n is regular of degree $2n$.

By the Handshake Lemma, the number of edges of H_n is

$$\frac{\sum_{v \in V(H_n)} deg(v)}{2} = \frac{2n|V(H_n)|}{2} = n3^n.$$