MATH 239 Quiz — Fall 2007 — SOLUTIONS

No calculators or other aids may be used. Each part-question is worth 5 marks.

1. (a) Determine the following coefficient, as a summation, where n and t are nonnegative integers.

$$[x^n](1-x^3)^t(1-4x)^{-4}$$

Solution

$$\begin{aligned}[x^n](1-x^3)^t(1-4x)^{-4} &= [x^n]\left(\sum_{k=0}^{t}\binom{t}{k}(x^3)^k\right)\left(\sum_{j\geq 0}\binom{4+j-1}{j}(4x)^j\right)\\ &= [x^n]\sum_{k=0}^{t}\sum_{j\geq 0}\binom{t}{k}\binom{3+j}{3}4^j x^{3k+j}.\end{aligned}$$

To achieve x^n we need $3k+j=n$, or $j=n-3k$. Then $0\leq j$ is equivalent to $3k\leq n$, or $k\leq\lfloor n/3\rfloor$. If $k>t$, the binomial coefficient $\binom{t}{k}$ is 0 in any case so the condition $k\leq t$ may be ignored. So the answer is

$$[x^n]\sum_{k=0}^{t}\sum_{j\geq 0}\binom{t}{k}\binom{3+j}{3}4^j x^{3k+j} = \sum_{k=0}^{\lfloor n/3\rfloor}\binom{t}{k}\binom{3+n-3k}{3}4^{n-3k}.$$

Note there are several alternative solutions. For instance we could sum over j, using $k=(n-j)/3$. But then only j congruent to n (mod 3) should be included in the summation.

(b) Let a_n be the number of compositions of n with an odd number of parts. Find the generating function $\sum_{n\geq 0} a_n x^n$, expressed as a rational function.

Solution. Let S be the set of all compositions with an odd number of parts. Then

$$\begin{aligned}S &= \mathbb{N}_{\geq 1}\cup {\mathbb{N}_{\geq 1}}^3\cup {\mathbb{N}_{\geq 1}}^5\cup\cdots\\ &= \bigcup_{k\geq 0}{\mathbb{N}_{\geq 1}}^{2k+1},\end{aligned}$$

where $\mathbb{N}_{\geq 1}=\{1,2,3,\ldots\}$. Note that

$$\Phi_{\mathbb{N}_{\geq 1}}(x) = x+x^2+x^3+\cdots = \frac{x}{1-x}.$$

Thus, by the sum and product lemmas,

$$\begin{aligned}\Phi_S(x) &= \sum_{k\geq 0}\Phi_{{\mathbb{N}_{\geq 1}}^{2k+1}}(x)\\ &= \sum_{k\geq 0}\left(\Phi_{\mathbb{N}_{\geq 1}}(x)\right)^{2k+1}\\ &= \sum_{k\geq 0}\left(\frac{x}{1-x}\right)^{2k+1}\\ &= \frac{\frac{x}{1-x}}{1-\left(\frac{x}{1-x}\right)^2}\\ &= \frac{x-x^2}{1-2x}\end{aligned}$$

2. (a) Give a decomposition of the binary strings in which every block of 1's that is followed by a block of 0's has length at least 3, such that these strings are uniquely created. Explain briefly why they are uniquely created.

Solution

The standard block decomposition is $\{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^*$. In this question, the blocks of 1's "inside" the decomposition are to have size at least 3. Thus, the $\{1\}\{1\}^*$ blocks should be replaced by $\{111\}\{1\}^*$ blocks. So the desired decomposition is

$$\{0\}^*(\{111\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^* .$$

The decomposition is unique since we are still using the standard block decomposition, but limiting the sizes of the blocks that can occur.

(b) For each of the following expressions, are the elements uniquely created? Explain.

i. $\{\{11, 111\}\{00\}\{111\}\{1\}^*\}^*$

ii. $\{\{00\}\{111\}\{1\}^*\}^*$

Solution

i. No. The string 110011111100111 can be created in two different ways:

$$11001111, 1100111$$
$$1100111, 11100111$$

Here the commas delimit successive elements of $\{11, 111\}\{00\}\{111\}\{1\}^*$.

ii. Yes. The decomposition is a block decomposition consisting of alternating blocks of 0's (namely, $\{00\}$) and 1's (namely, $\{111\}\{1\}^*$). Elements of a block decomposition are always uniquely created, as long as the elements of each constituent block are uniquely created. In this case, since the elements of $\{00\}$ and $\{111\}\{1\}^*$ are uniquely created, it follows that the elements of $\{\{00\}\{111\}\{1\}^*\}^*$ are also uniquely created.