MATH 239 — Fall 2007 – Sample Quiz — SOLUTIONS

No calculators or other aids may be used. Each part-question is worth 5 marks.

1. (a) Determine the following coefficient, as a summation, where t and n are nonnegative integers.

$$[x^n](1-x)^t(1+2x^2)^{-3}$$

[5]

Solution

$$\begin{aligned}
[x^n](1-x)^t(1+2x^2)^{-3} &= [x^n]\left(\sum_{k=0}^{t}\binom{t}{k}(-x)^k\right)\left(\sum_{j\geq 0}\binom{3+j-1}{j}(-2x^2)^j\right) \\
&= [x^n]\left(\sum_{k=0}^{t}\binom{t}{k}(-1)^k x^k\right)\left(\sum_{j\geq 0}\binom{2+j}{2}(-2)^j x^{2j}\right) \\
&= [x^n]\sum_{k=0}^{t}\sum_{j\geq 0}\binom{t}{k}(-1)^k\binom{2+j}{2}(-2)^j x^{k+2j}.
\end{aligned}$$

It is easiest to sum over $j \geq 0$ first. To achieve x^n we need $k+2j=n$, or $k=n-2j$. Then $0 \leq k$ is equivalent to $2j \leq n$, or $j \leq \lfloor n/2 \rfloor$. Also $k \leq t$ is equivalent to $n-2j \leq t$, but if $k > t$, the binomial coefficient $\binom{t}{k}$ is 0 in any case so this condition may be ignored. So the answer is

$$\begin{aligned}
[x^n]\sum_{j\geq 0}\sum_{k=0}^{t}\binom{t}{k}(-1)^k\binom{2+j}{2}(-2)^j x^{k+2j} &= \sum_{j=0}^{\lfloor n/2 \rfloor}\binom{t}{n-2j}(-1)^{n-2j}\binom{2+j}{2}(-2)^j \\
&= \sum_{j=0}^{\lfloor n/2 \rfloor}\binom{t}{n-2j}\binom{2+j}{2}(-1)^{n-j}2^j.
\end{aligned}$$

Note there are several alternative solutions. For instance the lower limit of the summation over j can be $\max\{0, \lceil (n-t)/2 \rceil\}$. A rather different alternative is to sum over k, using $j=(n-k)/2$. But then only k congruent to n (mod 2) should be in the summation.

(b) Find the generating function for compositions of n, where each part is an odd number greater than 4. (Note the number of parts is not fixed.) [5]

Solution Each individual part in such a composition is a member of $N_{\text{odd }>4} = \{5, 7, 9, \ldots\}$. Now

$$\begin{aligned}
\Phi_{N_{\text{odd }>4}}(x) &= x^5 + x^7 + x^9 + \cdots \\
&= x^5(1-x^2)^{-1}.
\end{aligned}$$

By the product lemma, the generating function for a k-part composition $N^k_{\text{odd }>4}$ is

$$\Phi_{N^k_{\text{odd }>4}}(x) = \left(x^5(1-x^2)^{-1}\right)^k.$$

By the sum lemma, the generating function for all compositions in question, $S = \cup_{k\geq 0} N^k_{\text{odd } >4}$, is

$$\begin{aligned}
\Phi_S(x) &= \sum_{k\geq 0} \Phi_{N^k_{\text{odd } >4}}(x) \\
&= \sum_{k\geq 0} \left(x^5(1-x^2)^{-1}\right)^k \\
&= \frac{1}{1-x^5(1-x^2)^{-1}}, \quad \text{by Geometric Series} \\
&= \frac{1-x^2}{1-x^2-x^5}, \quad \text{by multiplying top and bottom by } 1-x^2.
\end{aligned}$$

2. (a) Write a decomposition of the set S of binary strings that do not contain a block of 0's of length exactly 2, that uniquely creates the elements of S. [5]

Solution The standard 1-decomposition of all binary strings is

$$\{0\}^*(\{1\}\{0\}^*)^*.$$

The 0-blocks occur in the decomposition at each nonempty element of $\{0\}^*$. To make sure the blocks of 0's do not have length 2, we use $\{0\}^* \setminus \{00\}$ for them, or equivalently $\{\varepsilon\} \cup \{0\} \cup \{000\}\{0\}^*$. So the answer is

$$(\{0\}^* \setminus \{00\})\big(\{1\}(\{0\}^* \setminus \{00\})\big)^*.$$

Note Alternatively using the block decomposition

$$\{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^*$$

gives the valid solution

$$(\{0\}^* \setminus \{00\})\big(\{1\}\{1\}^*(\{0\}\{0\}^* \setminus \{00\})\big)^*\{1\}^*$$

and note here that the empty string ε is not possible in the 0-blocks in the repeating part of the decomposition.

(b) Find the generating function with respect to length for the set Q of binary strings, where

$$Q = (\{\varepsilon\} \cup \{00\})(\{1\}\{11\}^*\{00\}\{0\}^*)^*\{1\}^*.$$

Express your answer as $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials. [5]

Solution

$$\begin{aligned}
(1+x^2)&\frac{1}{1-x(1-x^2)^{-1}x^2(1-x)^{-1}}(1-x)^{-1} \\
&= \frac{1+x^2}{1-x-x^3(1-x^2)^{-1}} \\
&= \frac{(1+x^2)(1-x^2)}{(1-x)(1-x^2)-x^3} \\
&= \frac{1-x^4}{1-x-x^2}.
\end{aligned}$$

The second last line is OK for a final answer.