MATH 239 Assignment 7 Solutions

1. Find a maximum matching and a minimum cover in the graph in Figure 1.

   Solution:

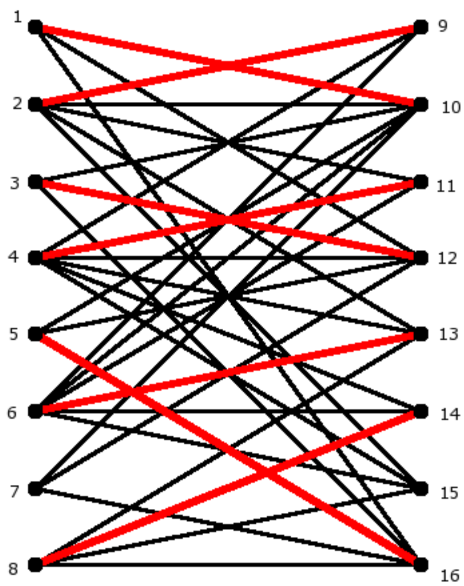

Figure 1: A maximum matching

We claim the matching shown in bold in Figure 1 is maximum. To show this, we find a cover of the same size. Following the algorithm on page 183 of the course notes, let M be the matching above, and $V = (A, B)$ with A being the vertices $\{1, 2, \ldots, 8\}$.

Step 1. Set $\hat{X} = \{7\}$, $\hat{Y} = \emptyset$.

Step 2. Let $\hat{Y} = \{10, 12, 16\}$ and set $pr(10) = pr(12) = pr(16) = 7$.

Step 3. Step 2 added some vertices to $\hat{Y}$, continue.

Step 4. No unsaturated vertices in $\hat{Y}$.

Step 5. Add $\{1, 3, 5\}$ to $\hat{X}$, and set $pr(1) = 10, pr(3) = 12$ and $pr(5) = 16$. Now $\hat{X} = \{1, 3, 5, 7\}$. Go to Step 2.

Step 2. No new vertices added to $\hat{Y}$.

Step 3. M is a maximum matching and the cover $C = \hat{Y} \cup (A \backslash \hat{X}) = \{2, 4, 6, 8, 10, 12, 16\}$ is minimum.

Indeed the maximum matching above and the minimum cover $C = \{2, 4, 6, 8, 10, 12, 16\}$ have the same size (7).

2. Find a subset D of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ such that $|N(D)| < D$.

   Solution: A suitable choice can be found from the proof of Hall's Theorem as $D = \hat{X}$: so the set $D = \{1, 3, 5, 7\}$ works. Its neighbourhood is $\{10, 12, 16\}$.

3. Let A be an $n \times n$ matrix. Formulate the problem of finding the largest set of non-sero entries, no two in the same row or column, as a matching problem in some bipartite graph. Interpret in terms of the matrix what a cover in the graph is.

   Solution: Let G be the bipartite graph with vertex classes $R = \{r_1, \ldots, r_n\}$ and $C = \{c_1, \ldots, c_n\}$, in which $r_i c_j$ is an edge precisely when the ij-entry of A is nonzero. Then a matching in G corresponds to a set of nonzero entries of A, no two of which are in the same row (otherwise they would both be incident to the same vertex of R), or the same column (similarly for C). Thus a maximum matching in G is a set of nonzero entries of maximum size, such that no two are in the same row or column.

   A cover in G is a set of rows and coluumns $R' \cup C'$ of A with the property that every nonzero entry of A is in a row of R' or a column of C' (or both).

4. Let k be a positive integer and suppose G is a bipartite graph in which every vertex has degree precisely k. Show:

   (a) any bipartition (A, B) of G has $|A| = |B|$
   Solution: Let (A, B) be a bipartition of G. Since every vertex has degree exactly k, by counting edges of G by their vertex in A we know that $|E(G)| = k|A|$. But similarly $|E(G)| = k|B|$. Therefore $k|A| = k|B|$ which implies $|A| = |B|$ (since $k \neq 0$).

   (b) G has a perfect matching
   Solution: We apply Hall's Theorem. Let D be a subset of A. Then the set E_D of edges incident to D has size exactly $k|D|$. Since $N(D)$ is by definition the set of vertices in B that are incident to an edge of E_D, and every vertex in B has degree exactly k, we know that $|E_D| \leq k|N(D)|$. Therefore

   $$k|D| = |E_D| \leq k|N(D)|,$$

   which implies $|D| \leq |N(D)|$. Therefore $|N(D)| \geq |D|$. Since this is true for every subset D of A, Hall's Condition holds and therefore G has a perfect matching.

   (c) G has k perfect matchings, no two having an edge in common.
   Solution: We use induction on k. If $k = 1$ then a 1-regular graph is exactly a perfect matching, so the claim holds.
   Suppose $k \geq 2$ and the claim holds for smaller values of k. By (b) we know G has a perfect matching M. Let G' be the graph obtained by removing the edges of M from G. Since M is a perfect matching, the degree of every vertex goes down by exactly one. So every vertex of G' has degree exactly $k-1$. By induction, G' has $k-1$ perfect matchings, no two of which share an edge. Then these together with M form k perfect matchings of G, no two of which share an edge.

5. For each positive integer n, find an example of a bipartite graph with n vertices on each side, with minimum degree at least three, and with no matching of size larger than $n/4$.

   (Note: in fact this claim is false unless $n/4 \geq 6$. So we give the solution only for $n \geq 24$.)

   Solution: Let $n \geq 12$ be given. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. The graph G is formed as follows: each $a \in A \setminus \{a_1, a_2, a_3\}$ is adjacent to $\{b_1, b_2, b_3\}$, and each $a \in \{a_1, a_2, a_3\}$ is adjacent to $B \setminus \{b_1, b_2, b_3\}$. Then each $a \in A$ and $b \in B$ has degree at least 3. Moreover, this graph has a cover of size 6, as every edge is incident to a vertex in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. Therefore by König's Theorem, the maximum size of a matching is at most 6. But if $n \geq 24$ then $6 \leq n/4$ so the claim holds.