

1. **[3 marks]** Show that the number of k -subsets of $\{1, \dots, n\}$ that contain 1 but do not contain n is $\binom{n-2}{k-1}$.
2. **[7 marks]** Let a_n be the number of compositions of n into three parts, where the first part is a multiple of 10, the second part is a multiple of 20 and the third part is a multiple of 50. Find an expression for the generating function $\sum_{n \geq 0} a_n x^n$ and show that it is equal to

$$\frac{x^{80}}{(1-x^{10})^2(1+x^{10})(1-x^{50})}$$

3. **[8 marks]**
 - (a) Write down a decomposition that uniquely creates the set of binary strings in which an odd-length block of 1's is never followed by an odd-length block of 0's.
 - (b) There is a set S of strings with generating function

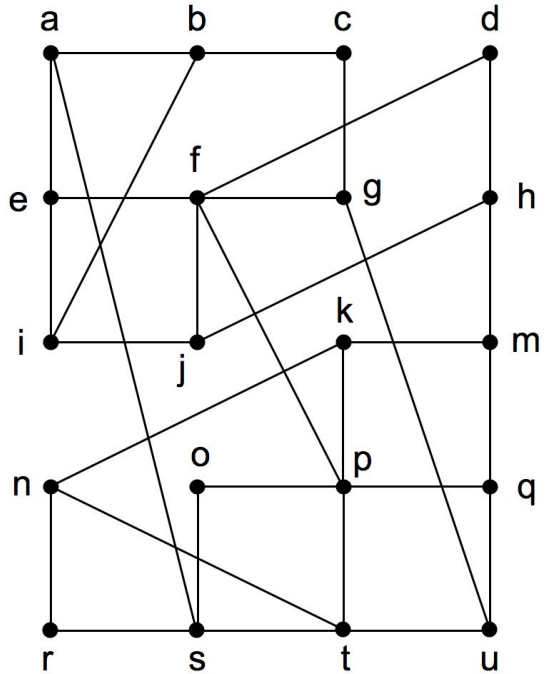
$$\Phi_S(x) = \frac{1+x}{1-x-x^2}$$

Let a_n be the coefficient of x^n in $\Phi_S(x)$. Find a recurrence relation for a_n , with enough initial conditions to uniquely determine the sequence of coefficients.

4. **[12 marks]** Solve the following homogeneous recurrence equation with initial conditions:

$$b_n - 5b_{n-1} + 8b_{n-2} - 4b_{n-3} = 0 \quad (n \geq 3); \quad b_0 = 2, b_1 = 2, b_2 = 0.$$

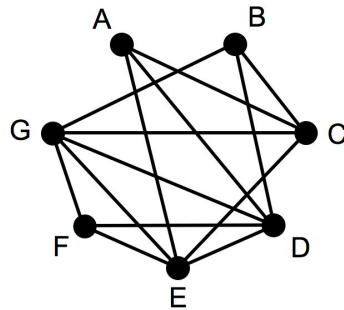
5. **[10 marks]**
 - (a) State the definition of a *tree*.
 - (b) Prove that every tree T with exactly 7 vertices of degree 1 and exactly 2 vertices of degree 4 must have exactly 1 vertex of degree 3.
 - (c) Draw a tree with 11 vertices in total, with at least 7 vertices of degree 1 and at least 2 vertices of degree 4.
6. **[12 marks]**
 - (a) Find a breadth first search tree in the graph shown, rooted at the vertex a . When considering the new vertices adjacent to a vertex in the tree, take them in increasing alphabetical order.



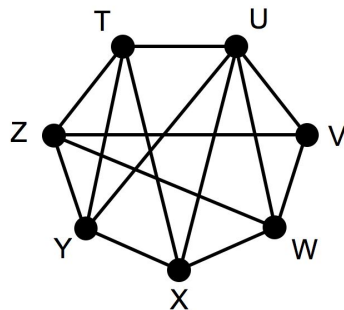
(b) Determine if the graph in (a) is bipartite. Give reasons for your answer. If it is bipartite, find a partition of the vertex set that shows it is bipartite. If not, find an odd cycle.

7. [8 marks] Prove that if a connected planar embedding has all faces of degree 3 and all vertices of degree 5, then it must have exactly 30 edges.
8. [10 marks] For each of the two graphs below, determine if it is planar. Prove your answer is correct in each case.

G_1 :



G_2 :



9. [5 marks] Prove that every planar graph with no 3-cycles is 4-colourable, assuming that every planar graph with no 3-cycles has a vertex of degree at most 3.

You may not assume that the 4-colour theorem is true, and you *do not* need to prove the assumption that there is a vertex of degree at most 3.

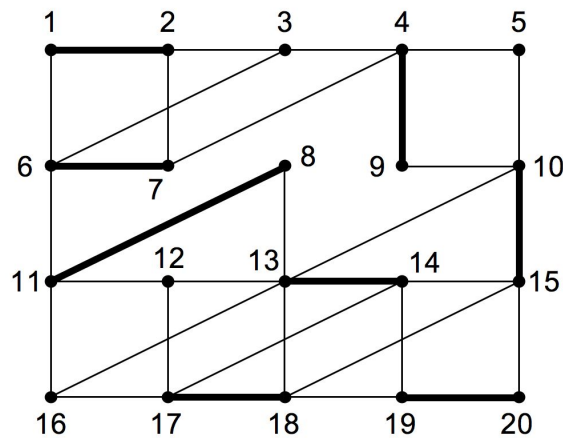
10. [8 marks]

- (a) State Hall's Theorem for when a bipartite graph with bipartition (A, B) has a matching saturating all vertices in A .
- (b) Let G be a bipartite graph with bipartition (A, B) such that $|A| = 20$, $|B| = 24$, and all vertices have degree at least 10. Show that $|N(D)| \geq |D|$ for all $D \subseteq A$ with $|D| \leq 10$. Using this, or otherwise, prove that G has matching saturating all vertices in A .

11. [12 marks] Let G be the bipartite graph below, with bipartition (A, B) where A consists of the vertices with odd numbers and B the even numbered ones.

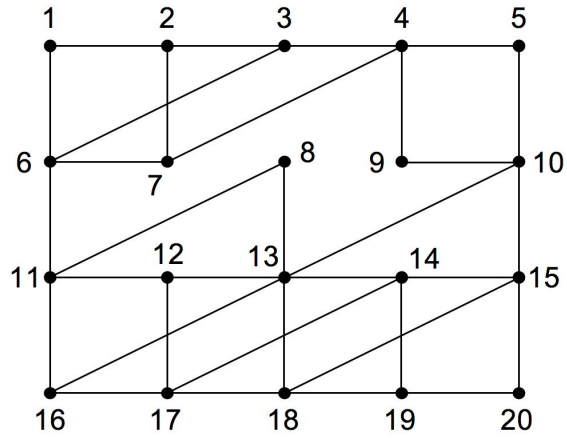
- (a) Beginning with the matching indicated with thick lines, apply the bipartite matching algorithm to find a larger matching.

Show the vertices of X and Y in the order that they are added to these two sets during the course of the algorithm.

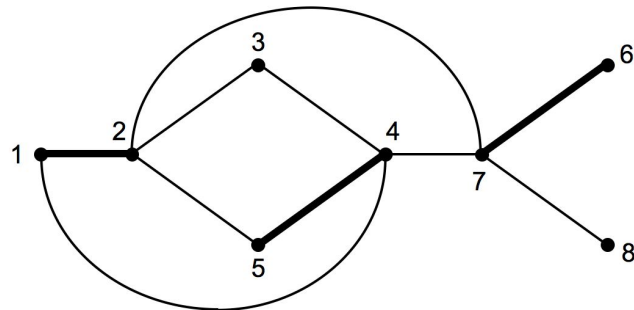


Vertices added
to X :
to Y :

Indicate the matching found:



- (b) Show the sets X and Y that are obtained if the XY construction is used in the following bipartite graph, where $A = \{1, 3, 5, 7\}$ and the starting matching is the one shown with thick lines. Hence give a minimum cover for the graph.



X :

Y :

Minimum cover: