

1. Proof required.

2.

$$\left(\frac{x^{10}}{1-x^{10}}\right)\left(\frac{x^{20}}{1-x^{20}}\right)\left(\frac{x^{50}}{1-x^{50}}\right)$$

3. (a)  $\{0\}^* (\{1\}\{1\}^*\{0\}\{0\}^* \setminus \{1\}\{11\}^*\{0\}\{00\}^*)^* \{1\}^*$

(b)  $a_n = a_{n-1} + a_{n-2}, \quad n \geq 2.$

$a_0 = 1, a_1 = 2.$

4.  $b_n = (2 - n)2^n.$

5. (a) a connected graph with no cycles.

(b) Let  $n_i$  be the number of vertices of degree  $i$ . Then using sum of degrees =  $2q$  (Handshake theorem), and also  $q = p - 1$  for trees, we can see that

$$n_1 - n_3 - 2n_4 - 3n_5 - \dots = -2.$$

This is true for all trees. Substituting  $n_1 = 7$  and  $n_4 = 2$  we get

$$7 - n_3 - 2 \times 4 - 3n_5 - \dots = -2.$$

Thus

$$1 = n_3 + 3n_5 + 4n_6 + \dots.$$

Since the  $n_i$  are nonnegative integers, the only possibility is  $n_3 = 1$  (and  $n_5 = n_6 = \dots = 0$ ).

(c) Should be easy.

6. (a) The vertices at levels are as follows (if these are correct, you probably have the right tree):

level 0: a

level 1: b e s

2: c i f o r t

3: g j d p n u

4: h k q

5: m

(b) The graph is not bipartite because there is an edge between two vertices at the same level (gu). Tracing back up the tree we find the cycle dfguqmhd.

7. Use the handshake theorem: sum of vertex degrees is  $5p = 2q$ .

Also the analogue for faces ('face-shaking'): sum of face degrees is  $3s = 2q$  (assuming  $s$  denotes the number of faces).

So  $p = 2q/5$  and  $s = 2q/3$ .

Plug these into Euler's theorem: since the embedding is connected,

$$p - q + s = 2$$

and this gives  $q = 30$ .

8. (a) is planar. (Draw it.)  
 (b) is not planar. It contains an edge subdivision of  $K_{3,3}$ .
9. Requires a proof. Similar to proof of 6 colour theorem.
10. (a) state Hall's theorem (Bipartite graph with bipartition  $A, B$  has matching saturating all vertices in  $A$  iff  $|N(D)| \geq |D|$  for all  $D \subseteq A$ .)  
 (b) Let  $D \subseteq A$ . If  $D = \emptyset$  then  $|N(D)| \geq |D|$  immediately. Next, every vertex has at least 10 neighbours. So every nonempty set  $D$  of vertices has  $|N(D)| \geq 10$ . Thus  $|N(D)| \geq |D|$  if  $|D| \leq 10$ . Now suppose  $|D| \geq 11$  for  $D \subseteq A$ . We show that  $|N(D)| = 24$ . If not, then there is a vertex  $b \in B$  such that  $b \notin N(D)$ . Then  $b$  is adjacent to no vertex in  $D$ . That is,  $N(b)$  is disjoint from  $D$ . But the graph is bipartite and  $b$  has degree at least 10, so at least 10 vertices in  $A$  are not in  $D$  (i.e. the vertices in  $N(b)$ ). Since  $|A| = 20$  this means  $|D|$  must be at most 10. (Contradiction). So  $|N(D)| = 24$  if  $|D| \geq 10$ . Since  $|D| \leq 20$  this implies  $|N(D)| \geq |D|$ , as required.
- Finally, since  $|N(D)| \geq |D|$  for all  $D \subseteq A$ , the graph has a matching saturating all vertices in  $A$  by Hall's theorem.
11. (a) Vertices added in the following order:  
 To X: 3,5  
 To Y: 2,4,6,10  
 To X: 1,9,7,15  
 To Y: 14, 18, 20  
 To X: 13, 17,19  
 To Y: 8, 12, 16  
 Stop: 12 is unsaturated.  
 Augmenting path is 5, 10, 15, 14 13, 12.  
 New matching has edges 1 2, 6 7, 4 9, 5 10, 15 14, 13 12, 8 11, 17 18, 19 20.
- (b) Minimum cover 2,4,7.