# Math 239 in a Nutshell

## Enumeration and Recurrence Relations

- Combinatorial proofs
- Binomial Theorem $(1+x)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} x^k$
- Generating functions
- Sum and product lemmas
  - application 1: compositions of an integer
  - application 2: (uniquely created) sets of binary strings, decomposition theorems
- Recurrence relations $\longleftrightarrow$ generating functions
- Solving recurrence relations explicitly

## Graph theory

- Basic definitions: isomorphism, walks, paths, connectedness, cuts, cycles, bridges
  - $\sum \deg(v) = 2q$
  - corollary: the number of vertices of odd degree is even
  - if there's a walk from $u$ to $v$, there's a path from $u$ to $v$
  - $G$ connected $\iff$ every proper nonempty $X \subset V(G)$ induces a nonempty cut
  - an edge is a bridge $\iff$ it's not in a cycle
  - $e = \{x, y\}$ a bridge $\iff$ $x, y$ in different components of $G - e$
  - two distinct paths from $u$ to $v \implies G$ contains a cycle
- Trees
  - unique path between any two vertices
  - $q = p - 1$
  - at least two leaves (vertices of degree 1)
- Spanning Trees, BFSTs, girth
  - exist $\iff$ $G$ is connected
  - algorithm for growing a BFST
  - primary property of BFSTs: adjacent vertices are at most 1 level apart
  - application 1: distance from $x$ to root = level($x$)
  - application 2: bipartite $\iff$ no odd cycles
  - application 3: $\text{girth}(G) \leq \text{level}(x) + \text{level}(y) + 1$ for $\{x, y\} \in E(G) \setminus T$
- Planar graphs, planar embeddings
  - fundamental fact (hard): $e$ is a bridge $\iff$ it is incident with exactly 1 face
  - $\sum \deg(f_i) = 2q$
  - Euler's formula: $p - q + s = 1 + c$
  - $q \leq \frac{d^*(p-2)}{d^*-2}$, where $d^* \leq \deg(f_i)$
  - special case: $d^* = \text{girth}(G)$, or $d^* \leq \text{girth}(G)$
  - Kuratowski's Theorem: $G$ nonplanar $\iff G$ contains an edge subdivision of $K_5$ or $K_{3,3}$
  - every planar graph has a vertex of degree $\leq 5$
  - application: 6-colour theorem
- Matchings, covers
  - augmenting path can be used to produce a larger matching
  - covers are at least as big as matchings: $|C| \geq |M|$.
  - corollary: if $|C| = |M|$ then $C$ is a min. cover, and $M$ is a max. matching
  - König's theorem: if $G$ bipartite, max. matchings and min. covers are the same size
  - algorithm for finding maximum matchings
  - Hall's theorem, systems of distinct representatives
  - application: $k$-regular bipartite graphs have perfect matchings